Consider the continuous-time state-space equations:

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{1}$$
$$y(t) = Cx(t) + Du(t). \tag{2}$$

One simple way to do discretization is that we apply Euler method, i.e.,

$$x(t+T) = x(t) + Ax(t)T + Bu(t)T,$$

where T is the time interval for discretization. Then at time $t = kT$, we have

$$x\left((k+1)T\right) = (I + TA)x(kT) + TBu(kT),$$
$$y(kT) = Cx(kT) + Du(kT),$$

where I represents identity matrix. This discretization is easy but not accurate especially when T is large, so we consider a different discretization.

For the discretization method to be discussed next, an assumption is made first. That is, in each time interval T, the input is assumed to be constant. This is not a strong assumption as this situation is very common in computer systems. Based on the assumption, we have for $kT \le t < (k+1)T$,

$$u(t) = u(kT) =: u[k].$$

For (1), its analytical solution at $t = kT$ and $t = (k+1)T$ is, respectively, [1]

$$x[k] = x(kT) = e^{AkT}x(0) + \int_0^{kT} e^{A(kT-\tau)}Bu(\tau)d\tau,$$

and

$$x[k+1] = x\left((k+1)T\right) = e^{A(k+1)T}x(0) + \int_0^{(k+1)T} e^{A((k+1)T-\tau)}Bu(\tau)d\tau,$$

which can be further written as

$$\begin{aligned} x[k+1] &= e^{AT}\left[e^{AkT}x(0) + \int_0^{kT} e^{A(kT-\tau)}Bu(\tau)\right] + \int_{kT}^{(k+1)T} e^{A(kT+T-\tau)}Bu(\tau)d\tau, \\ &= e^{AT}x[k] + \left(\int_{kT}^{(k+1)T} e^{A(kT+T-\tau)}d\tau\right)Bu[k]. \end{aligned}$$

By introducing a new variable $\alpha := kT + T - \tau$, we have

$$x[k+1] = e^{AT}x[k] + \left(\int_0^T e^{A\alpha}d\alpha\right)Bu[k].$$

Therefore, (1) and (2) at time $t = kT$ become

$$x[k+1] = A_d x[k] + B_d u[k], \tag{3}$$
$$y[k] = C_d x[k] + D_d u[k]. \tag{4}$$

with

$$
\begin{aligned}
A_d &= e^{AT}, \\
B_d &= \left(\int_0^T e^{A\tau} d\tau \right) B, \\
C_d &= C, \\
D_d &= D.
\end{aligned}
$$

In this discretization there is no approximation involved, and (4) is the exact solution of (1) at $t = kT$ if the input is piecewise constant.

References

[1] C.-T. Chen, *Linear System Theory and Design.* Oxford University Press, Inc., 1998.